Title: Velocity dependent Coulomb logarithm in the Landau limit of the Boltzmann equation

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Velocity dependent Coulomb logarithm in the Landau limit of the Boltzmann equation

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Summary

Introduction

The Fokker-Planck-Landau (FPL) operator is an approximation of the Boltzmann equation :

$$Q_L(f,f) = \log\Lambda\, \nabla_{\boldsymbol{v}} \cdot \left(\int_{\mathbb{R}^3} |\boldsymbol{u}|^2 \left(I - \frac{\boldsymbol{u}\otimes\boldsymbol{u}}{|\boldsymbol{u}|^2} \right) \left(f(\boldsymbol{v}_*)\nabla_{\boldsymbol{v}} f(\boldsymbol{v}) - f(\boldsymbol{v})(\nabla_{\boldsymbol{v}} f)(\boldsymbol{v}_*) \right) d\boldsymbol{v}_* \right), \tag{1}$$

where $\log\Lambda$ is the *Coulomb Logarithm* (CL).

- This approximation is valid in the limit where grazing collisions dominate the collision process, which occurs due to the long range nature of the Coulomb interaction potential.
- Generally considered valid when $\log\Lambda \approx 10$–20, regime of weakly coupled plasmas.

Motivation

Motivation :

- The CL has been derived before but it is unclear when/where to use it
- Jeff Haack and Irene Gamba have worked on this previously but in a more mathematical framework
- The CL is usually assumed to be a constant but in principle a velocity dependent CL can be derived from Boltzmann
- Direct numerical comparisons between Boltzmann and FPL are difficult so it's not clear how much this may matter
- In this talk we will use the spectral formulation of Boltzmann to find a consistent spectral formulation of the FPL using the velocity dependent CL
- Would using a velocity dependent CL change the $\mathcal{O}(1)$ term between Boltzmann and FPL ?

$$Q_B = log\Lambda Q_L + \mathcal{O}(1) \tag{2}$$

Boltzmann equation

The space homogeneous Boltzmann equation is given by

$$\frac{\partial}{\partial t} f(\boldsymbol{v}, t) = Q_B(f, f)(\boldsymbol{v}, t), \tag{3}$$

with

$$f(\boldsymbol{v}, 0) = f_0(\boldsymbol{v}) \text{ and } \boldsymbol{v} \in \mathbb{R}^3 \tag{4}$$

and

- $f(\boldsymbol{v}, t)$ is a probability density function
- $f_0(\boldsymbol{v})$ is the initial condition
- $Q(f, f)$ is given by the bilinear integral form

Integral Form

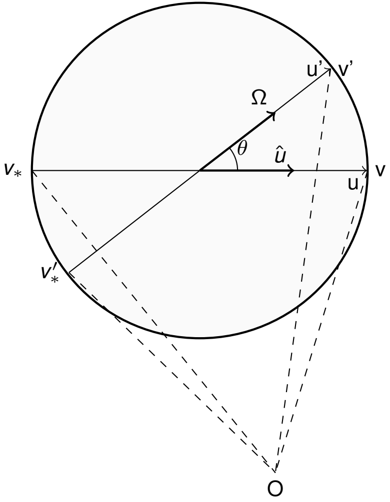
$$Q_B(f,f)(\boldsymbol{v},t) = \int_{\mathbb{R}^3}\int_{S^2} |\boldsymbol{u}|\sigma(|\boldsymbol{u}|,\cos\theta)\left(f(\boldsymbol{v}_*')f(\boldsymbol{v}') - f(\boldsymbol{v}_*)f(\boldsymbol{v})\right) d\boldsymbol{\Omega} d\boldsymbol{v}_*. \tag{5}$$

- $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_*$ is the relative velocity
- $\boldsymbol{\Omega}$ is the scattering direction
- θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{\Omega}$
- $\sigma(|\boldsymbol{u}|,\cos\theta)$ is the differential cross section

The elastic post collisional velocities $\boldsymbol{v}'$, $\boldsymbol{v}_*'$ are given by

$$\boldsymbol{v}' = \boldsymbol{v} + \frac{1}{2}(|\boldsymbol{u}|\boldsymbol{\Omega} - \boldsymbol{u}), \qquad \boldsymbol{v}_*' = \boldsymbol{v}_* - \frac{1}{2}(|\boldsymbol{u}|\boldsymbol{\Omega} - \boldsymbol{u}). \tag{6}$$

Rotation of relative velocity



$$\boldsymbol{v}' = \boldsymbol{v} + \frac{1}{2}(|\boldsymbol{u}|\boldsymbol{\Omega} - \boldsymbol{u}), \qquad \boldsymbol{v}'_* = \boldsymbol{v}_* - \frac{1}{2}(|\boldsymbol{u}|\boldsymbol{\Omega} - \boldsymbol{u}).$$

Weak form of the collision operator

The weak form of the collision operator :

$$\int_{\mathbb{R}^3} Q_B(f,f)\phi(\boldsymbol{v})d\boldsymbol{v} = \int_{\mathcal{R}} f(\boldsymbol{v})f(\boldsymbol{v}_*)|\boldsymbol{u}|\sigma(|\boldsymbol{u}|,\cos\theta)\left(\phi(\boldsymbol{v}') - \phi(\boldsymbol{v})\right) d\boldsymbol{\Omega} d\boldsymbol{v}_* d\boldsymbol{v}, \quad (7)$$

where $\mathcal{R} = \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. If we take a test function $\phi(\boldsymbol{v})$ of the form

$$\phi(\boldsymbol{v}) = (2\pi)^{-3/2} e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{v}},$$

then we get

$$\widehat{Q_B}(\boldsymbol{\zeta}) = \int_{\mathbb{R}^3} \widehat{G}(\boldsymbol{\xi},\boldsymbol{\zeta})\hat{f}(\boldsymbol{\zeta}-\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}. \quad (8)$$

Here $G(\boldsymbol{u},\boldsymbol{\zeta})$ is defined as

$$G(\boldsymbol{u},\boldsymbol{\zeta}) = (2\pi)^{-3/2}|\boldsymbol{u}| \int_{S^2} \sigma(|\boldsymbol{u}|,\cos\theta)\left(e^{-i\frac{\boldsymbol{\zeta}}{2}\cdot(|\boldsymbol{u}|\boldsymbol{\Omega}-\boldsymbol{u})} - 1\right) d\boldsymbol{\Omega}, \quad (9)$$

The convolution weights $\widehat{G}$ are given by

$$\widehat{G}(\boldsymbol{\xi},\boldsymbol{\zeta}) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}|\boldsymbol{u}|e^{-i\boldsymbol{\xi}\cdot\boldsymbol{u}}\int_{S^2}\sigma(|\boldsymbol{u}|,\cos\theta)\left(e^{-i\frac{\boldsymbol{\zeta}}{2}\cdot(|\boldsymbol{u}|\boldsymbol{\Omega}-\boldsymbol{u})}-1\right)d\boldsymbol{\Omega}d\boldsymbol{u}. \quad (10)$$

After some math...

$$\widehat{G}(\boldsymbol{\xi},\boldsymbol{\zeta}) = (2\pi)^{1/2}\int_0^\infty r^3\int_0^\pi\int_0^\pi \sigma(r,\cos\theta)\sin\theta\sin\gamma J_0\left(r\left|\boldsymbol{\xi}-\frac{\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|^2}\boldsymbol{\zeta}\right|\sin\gamma\right)$$
$$\times\left[\cos\left(r\left(\boldsymbol{\xi}-\frac{\boldsymbol{\zeta}}{2}(1-\cos\theta)\right)\cdot\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}\cos\gamma\right)J_0\left(\frac{1}{2}r|\boldsymbol{\zeta}|\sin\gamma\sin\theta\right)\right.$$
$$\left.-\cos\left(r\xi\cdot\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}\cos\gamma\right)\right]d\theta d\gamma dr, \quad (11)$$

where $J_0(x)$ is the Bessel function of the first kind.

The convolution weights (11) **do not depend on time and can be precomputed to high accuracy using numerical integrators**

With a similar weak form of the FPL operator

$$\int_{\mathbb{R}^3} Q_L(f,f)\phi(\boldsymbol{v})d\boldsymbol{v} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \log\Lambda\, f(\boldsymbol{v})f(\boldsymbol{v}_*) \times \left(-4|\boldsymbol{u}|^{-3}\boldsymbol{u}\cdot\nabla\phi + |\boldsymbol{u}|^{-1}\left(I - \frac{\boldsymbol{u}\otimes\boldsymbol{u}}{|\boldsymbol{u}|^2}\right) : \nabla^2\phi\right) d\boldsymbol{v}d\boldsymbol{v}_*$$

where : denotes the matrix double dot product and ∇^2 denotes the Hessian.
We also take ϕ to be the Fourier basis function to get

$$\widehat{Q_L}(\zeta) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathcal{F}[f(\boldsymbol{v})f(\boldsymbol{v}-\boldsymbol{u})](\zeta)\, G_L(\boldsymbol{u},\zeta)d\boldsymbol{u}. \tag{12}$$

And $G_L(\boldsymbol{u},\zeta)$ is given by

$$G_L(\boldsymbol{u},\zeta) = \log\Lambda\,|\boldsymbol{u}|^{-3}\Big(4i(\boldsymbol{u}\cdot\zeta) - |\boldsymbol{u}|^2|\zeta^{\perp}|^2\Big), \tag{13}$$

where $\boldsymbol{\zeta}^{\perp} = \boldsymbol{\zeta} - \frac{\boldsymbol{\zeta}\cdot\boldsymbol{u}}{|\boldsymbol{u}|^2}\boldsymbol{u}$ is the orthogonal component of $\boldsymbol{\zeta}$ wrt $\boldsymbol{u}$.

Rutherford Cross Section

The Rutherford cross section is given by

$$\sigma(|\boldsymbol{u}|, \theta) = \left(\frac{Z^2 e^2}{8\pi\epsilon_0 m |\boldsymbol{u}|^2 \sin^2(\theta/2)} \right)^2 \tag{14}$$

where

- θ is the scattering angle
- Z is the charge state of the particles
- e is the elementary charge
- ϵ_0 is the vacuum permittivity
- m is the mass of the particle

Directly using this cross section in the Boltzmann collision operator results in a logarithmic singularity

Resolving the Singularity

The Rutherford cross section can be derived from the the scattering angle of a two body Coulomb interaction :

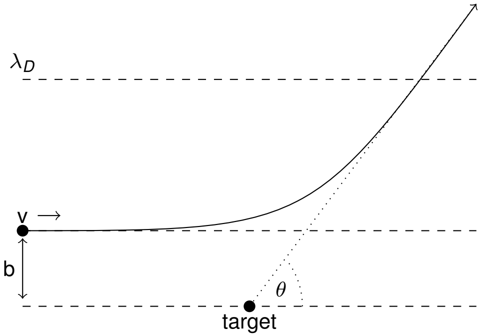
$$\theta(b, |\boldsymbol{u}|) = 2 \arctan\left(\frac{Z^2 e^2}{4\pi\epsilon_0 m |\boldsymbol{u}|^2 b}\right), \tag{15}$$

where b is the impact parameter. The differential cross section is defined through

$$\sigma(|\boldsymbol{u}|, \theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \tag{16}$$

Inverting equation (15) to solve for b, calculating the derivative of $b(|\boldsymbol{u}|, \theta)$ with respect to θ, and plugging into (16) we obtain the Rutherford cross section.

Coulomb Interaction



Velocity Dependent CL

Now, we note that charged particles are screened from one another at the Debye length λ_D. Thus, we cut off the impact parameter b at λ_D in (15), which corresponds to an angular cutoff at

$$\theta_m(|\boldsymbol{u}|) = \theta_m(\lambda_D, |\boldsymbol{u}|) = 2\arctan\left(\frac{Z^2e^2}{4\pi\epsilon_0 m|\boldsymbol{u}|^2\lambda_D}\right). \tag{17}$$

The differential cross section with cutoff is given by

$$\sigma(|\boldsymbol{u}|, \theta) = \left(\frac{Z^2e^2}{8\pi\epsilon_0 m|\boldsymbol{u}|^2\sin^2(\theta/2)}\right)^2 \mathbb{1}_{\theta>\theta_m(|\boldsymbol{u}|)}. \tag{18}$$

For the purposes of the analysis we rescale by the following

$$\sigma_{\theta_m}(|\boldsymbol{u}|, \theta)\sin\theta d\theta = -\frac{1}{2\pi\log(\sin(\theta_m/2))}\sigma(|\boldsymbol{u}|, \theta)\sin\theta\mathbb{1}_{\theta\geq\theta_m}d\theta. \tag{19}$$

Previously, the cutoff was a small parameter ϵ which was not velocity dependent.

Main Theorem

Theorem

Assume that f_{θ_m} satisfies

$$|\mathcal{F}\{f_{\theta_m}(\boldsymbol{v})f_{\theta_m}(\boldsymbol{v}-\boldsymbol{u})\}(\zeta)| \leq \frac{A(\zeta,t)}{1+|\boldsymbol{u}|^{3+a}} \tag{20}$$

with $A(\zeta,t)$ uniformly bounded by $k(1+|\zeta|)^{-3}$, k constant, and $a>0$. Then

$$||\widehat{Q_L}[f_{\theta_m}] - \widehat{Q_B}[f_{\theta_m}]|| \rightarrow_{\lambda_D\rightarrow\infty} 0 \tag{21}$$

and the error is

$$||\widehat{Q_L}[f_{\theta_m}] - \widehat{Q_B}[f_{\theta_m}]|| \leq \mathcal{O}\left(\frac{\left|1-\sin^2(\theta_m/2)\right|}{\log(\sin(\theta_m/2))}\left(\frac{|\zeta|^2}{|\boldsymbol{u}|}+|\zeta|^3\right)\right) \tag{22}$$

Proof of Theorem Use Taylor expansion on the exponential term and define σ in terms θ and ϕ with $\sigma = \cos\theta \frac{\boldsymbol{u}}{|\boldsymbol{u}|} + \sin\theta\omega$, where $\omega \in \mathcal{S}^1$:

$$\begin{aligned}
G_B(\boldsymbol{u}, \zeta) &= (2\pi)^{-3/2}|\boldsymbol{u}| \int_{S^2} \sigma_{\theta_m}(\hat{u}\cdot\boldsymbol{\Omega}) \left(e^{-i\frac{\zeta}{2}\cdot(|\boldsymbol{u}|\boldsymbol{\Omega}-\boldsymbol{u})} - 1\right) d\boldsymbol{\Omega} \\
&= (2\pi)^{-3/2}|\boldsymbol{u}| \int_0^{\pi}\int_0^{2\pi} \sigma_{\theta_m}(\cos\theta)\sin\theta \\
&\times \Bigg[i\left((\boldsymbol{u}\cdot\zeta)\sin^2(\theta/2) - |\boldsymbol{u}||\zeta^{\perp}|\sin(\theta/2)\cos(\theta/2)\sin\phi\right) \\
&- \frac{1}{2}\left((\boldsymbol{u}\cdot\zeta)\sin^2(\theta/2) - |\boldsymbol{u}||\zeta^{\perp}|\sin(\theta/2)\cos(\theta/2)\sin\phi\right)^2 \\
&- \frac{ie^{ic}}{6}\left((\boldsymbol{u}\cdot\zeta)\sin^2(\theta/2) - |\boldsymbol{u}||\zeta^{\perp}|\sin(\theta/2)\cos(\theta/2)\sin\phi\right)^3 \Bigg] d\phi d\theta \\
&:= G_{B_1} + G_{B_2} + G_{B_3}
\end{aligned} \tag{23}$$

for some c within $0 < |c| < \left|\frac{\boldsymbol{u}\cdot\zeta}{2} - |\boldsymbol{u}|\frac{\zeta\cdot\boldsymbol{\Omega}}{2}\right|$.

Takeaway : Exponential term will yield $1 - sin^2(\theta/2)$ term

We split up the computation into two lemmas in the following way

$$G_B = \underbrace{G_{B_1} + G_{B_2}}_{G_{B_1^r} + G_{B_2^r} + G_L} + G_{B_3}$$

Lemma 1 :

$$G_{B_1^r} + G_{B_2^r} \leq \mathcal{O}\left(\frac{|\boldsymbol{u}|^{-1}|\zeta|^2\left|1-\sin^2(\theta_m/2)\right|}{|\log(\sin(\theta_m/2))|}\right) \tag{24}$$

Lemma 2 :

$$G_{B_3} \leq \mathcal{O}\left(\frac{|\zeta|^3\left|1-\sin^2(\theta_m/2)\right|}{\left|\log(\sin^2(\theta_m/2))\right|}\right) \tag{25}$$

Lemma 1 Proof

Simplifying using trigonometric identities and applying the Fundamental Theorem of Calculus,

$$G_{B_1} + G_{B_2} = \frac{(2\pi)^{-3/2} C_1 |\boldsymbol{u}|^{-3}}{\log(\sin(\theta_m/2))} \Big(4i(\boldsymbol{u}\cdot\zeta)(\log(\sin(\theta_m/2)) + 2(\boldsymbol{u}\cdot\zeta)^2(1 - \sin^2(\theta_m/2)) - |\boldsymbol{u}|^2|\zeta^\perp|^2 \Big(\log(\sin(\theta_m/2)) + 1 - \sin^2(\theta_m/2) \Big) \Big). \tag{26}$$

Rearranging and recalling the weight function for the Landau operator, $G_L(\boldsymbol{u},\zeta) = |\boldsymbol{u}|^{-3}\Big(4i(\boldsymbol{u}\cdot\zeta) - |\boldsymbol{u}|^2|\zeta^\perp|^2\Big)$, we have

$$\begin{aligned} G_{B_1} + G_{B_2} &= G_L(\boldsymbol{u},\zeta) + \frac{(2\pi)^{-3/2} C_1 |\boldsymbol{u}|^{-3}\left(2(\boldsymbol{u}\cdot\zeta)^2 - |\boldsymbol{u}|^2|\zeta^\perp|^2\right)\Big(1 - \sin^2(\theta_m/2)\Big)}{\log(\sin(\theta_m/2))} \\ &= G_L(\boldsymbol{u},\zeta) + \frac{(2\pi)^{-3/2} C_1 |\boldsymbol{u}|^{-1}|\zeta|^2\Big(2\cos^2\alpha - \sin^2\alpha\Big)\Big(1 - \sin^2(\theta_m/2)\Big)}{\log(\sin(\theta_m/2)).} \end{aligned} \tag{27}$$

To clean up this calculation, let us define $G_{B_1^r}$ and $G_{B_2^r}$ as the leftover terms on the RHS of (27) and then taking the norm we obtain,

$$\begin{aligned}\left|G_{B_1^r} + G_{B_2^r}\right| &\leq \frac{(2\pi)^{-3/2} C_1 |\boldsymbol{u}|^{-1} |\zeta|^2 \left|2\cos^2\alpha - \sin^2\alpha\right| \left|1 - \sin^2(\theta_m/2)\right|}{|\log(\sin(\theta_m/2))|} \\ &\leq \frac{2(2\pi)^{-3/2} C_1 |\boldsymbol{u}|^{-1} |\zeta|^2 \left|1 - \sin^2(\theta_m/2)\right|}{|\log(\sin(\theta_m/2))|} \end{aligned} \quad (28)$$

Thus we have

$$G_{B_1} + G_{B_2} \leq G_L(\boldsymbol{u}, \zeta) + \mathcal{O}\left(\frac{|\boldsymbol{u}|^{-1}|\zeta|^2 \left|1 - \sin^2(\theta_m/2)\right|}{|\log(\sin(\theta_m/2))|}\right) \quad (29)$$

■

Lemma 2 Proof

Using trigonometric identities and integrating over ϕ we have,

$$G_{B_3} = \frac{i(2\pi)^{-3/2}|\zeta|^3|\boldsymbol{u}|^4}{12\log(\sin(\theta_m/2))}\int_{\theta_m}^{\pi}\sigma(\cos\theta)\sin\theta e^{ic}\sin^4(\theta/2)\cos\alpha \times\left(2\cos^2\alpha\sin^2(\theta/2)+3\sin^2\alpha\cos^2(\theta/2)\right)d\theta \quad (30)$$

Taking the norm we have and using the variable change $x=\sin(\theta/2)$ we obtain,

$$\begin{aligned}|G_{B_3}| &\leq \frac{5(2\pi)^{-3/2}|\zeta|^3|\boldsymbol{u}|^4}{12|\log(\sin(\theta_m/2))|}\int_{\theta_m}^{\pi}\sigma(\cos\theta)\sin\theta\sin^4(\theta/2)dx\\ &= \frac{5(2\pi)^{-3/2}C_1|\zeta|^3}{12|\log(\sin(\theta_m/2))|}\int_{\sin(\theta_m)}^{1}xdx\\ &= \frac{5(2\pi)^{-3/2}C_1|\zeta|^3|1-\sin^2(\theta_m/2)|}{12|\log(\sin(\theta_m/2))|}\end{aligned} \quad (31)$$

Thus, we have a control on G_{B_3},

$$G_{B_3} \leq \mathcal{O}\left(\frac{|\zeta|^3\left|1-\sin^2(\theta_m/2)\right|}{|\log(\sin^2(\theta_m/2))|}\right) \quad (32)$$

■

We can condense the above weights to

$$\begin{aligned}\tilde{G}(\boldsymbol{u},\zeta) &:= G_B(\boldsymbol{u},\zeta) - G_L(\boldsymbol{u},\zeta) \\ &= G_{B_1^r}(\boldsymbol{u},\zeta) + G_{B_2^r}(\boldsymbol{u},\zeta) + G_{B_3}(\boldsymbol{u},\zeta).\end{aligned} \tag{33}$$

Thus, one obtains,

$$\widehat{Q_B}[f_{\theta_m}](\zeta) - \widehat{Q_L}[f_{\theta_m}](\zeta) = \int_{\mathbb{R}^3} \mathcal{F}\{f_{\theta_m}(\boldsymbol{v})f_{\theta_m}(\boldsymbol{v}-\boldsymbol{u})\}(\zeta)\tilde{G}(\boldsymbol{u},\zeta)d\boldsymbol{u}. \tag{34}$$

Putting together (24) and (25), we obtain the final estimate

$$\left|\widehat{Q_B}[f_{\theta_m}](\zeta) - \widehat{Q_L}[f_{\theta_m}]\right| \leq$$

$$\left|\int_{\mathbb{R}^3} \mathcal{F}\{f_{\theta_m}(\boldsymbol{v})f_{\theta_m}(\boldsymbol{v}-\boldsymbol{u})\}\mathcal{O}\left(\frac{\left|1-\sin^2(\theta_m/2)\right|}{|\log(\sin(\theta_m/2))|}\left(\frac{|\zeta|^2}{|\boldsymbol{u}|} + |\zeta|^3\right)\right)d\boldsymbol{u}\right| \tag{35}$$

Recalling the definition of θ_m

$$\theta_m(|\boldsymbol{u}|) = \theta_m(\lambda_D, |\boldsymbol{u}|) = 2\arctan\left(\frac{Z^2 e^2}{4\pi\epsilon_0 m|\boldsymbol{u}|^2\lambda_D}\right), \tag{36}$$

and letting $r = |\boldsymbol{u}|$, then the estimate becomes

$$\left|\widehat{Q_B}[f_{\theta_m}](\zeta) - \widehat{Q_L}[f_{\theta_m}]\right| \leq$$

$$\int_0^\infty \frac{\sqrt{2}(1+|\zeta|)^{-3}}{(1+r^{3+a})} \frac{1}{\log\left(1+\frac{r^4\lambda_D^2}{C_1^2}\right)} \left(\frac{1}{1+\frac{C_1^2}{r^4\lambda_D^2}}\right)\left(|\zeta|^2 r + |\zeta|^3 r^2\right) dr \tag{37}$$

Note that as $r \to 0$, the integrand is finite.
Similarly, when $r \to \infty$, the integrand is finite.

Finally taking the limit as $\lambda_D \to \infty$, then the we have shown that the Landau operator Q_L and the Boltzmann operator Q_B converges to zero in the L^∞ difference.

Future Work

- Work on implementing these results in the numerical code
- Consistent comparison of Boltzmann and FPL operator with velocity dependent Coulomb logarithm
- Derive the next order correction for FPL using spectral method